

Legendre Matrix Method for Legendre Curve in Sasakian 3-Manifold

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Abstract. In this study, unit-speed the Legendre curves are studied in Sasakian 3-manifold. Firstly, differential equations characterizing the Legendre curves are obtained and the method used for the approximate solution is explained. Then, the approximate solution is found for one of the characterizations of the Legendre curve with the Legendre matrix collocation method. In addition, a sample application is made to make the method more understandable. And finally, with the help of these equations and the approximate solution, the geometric properties of this curve type are examined.

Keywords: Legendre curves, Contact manifold, Legendre matrix method, Sasakian 3-Manifold

1. Introduction

Linear differential equations with variable coefficients or systems of equations have an important place in differential geometry. The curves, one of the main topics of differential geometry, are characterized by a number of differential equations in different spaces. Although it isn't always possible to obtain analytical solutions of these differential equations, approximate solutions can be obtained with different methods [8,9,11]. Approximate solutions of the differential equation or system that characterizes a curve allow the geometric properties of the curve to be examined.

The Legendre curves are smooth plane curves which may have singular points, but still have a well defined smooth normal vector field [2]. In 3-dimensional contact geometry, the Legendre curves play a fundamental role [1]. The Legendre curves are defined as custom contact slant curves with a $\pi/2$ contact angle in the Sasakian 3-manifold, which is a special contact space. The unit 3-sphere S^3 is a typical example of a Sasakian manifold [14]. There are many studies on the Legendre curves in different spaces [15]:

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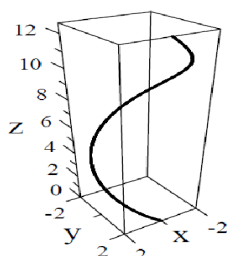


Figure 1. A Legendre curve in 3-dimensional BCV spaces

The Legendre curves are analyzed in 3-dimensional Sasaki spaces and some original results are obtained [6]. The spherical characterization of the Legendre curves in 3-dimensional quasi-Sasaki pseudo-metric manifolds is presented [13]. However, a study about the solution of differential equations characterizing the Legendre curves in Sasakian 3-manifold isn't available in the literature [7].

In order to obtain a characterization of a particular curve in any space, the first step is to create the equations for the roof to be used in that space. Baikoussis and Blair gave Frenet's equations for the Legendre curves in three-dimensional Sasaki space [1]. In 2002, Belkelfa et al. also gave Frenet equations of a Legendre curve in Lorentz geometry for three-dimensional Sasaki space [4]. In the 3-dimensional Sasaki space, finding Serret-Frenet equations for curves other than the Legendre curves is a difficult problem. Even Frenet equations for a Legendre curve in Sasaki spaces of different dimensions haven't been found yet.

In this study, firstly differential equations characterizing the unit-speed Legendre curves are obtained in Sasakian-3 manifold. On the other hand, a Legendre matrix collocation method based on the Legendre polynomials is developed. Then, the approximate solutions of the differential equations obtained are found with the help of this method.

2. Preliminaries

In this section, some basic definitions and concepts related to our study are discussed.

Let M be a smooth manifold. A contact form η on M is 1-form if $(\partial\eta)^n \wedge \eta \neq 0$. A manifold M together with a contact form is called a contact manifold [7, 13]. The ζ real vector field (killing vector field) is defined as $\eta(\zeta)=1$, $\partial\eta(\zeta,.)=0$. On a contact manifold (M, η) , there exist an endomorphism field ϕ and a Riemannian metric g satisfying

$$\begin{aligned}\phi^2 &= -I + \eta \otimes \zeta, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \partial\eta(X, Y) &= 2g(X, \phi Y)\end{aligned}$$

for all X and Y vector fields on M . The tensor structure (ζ, ϕ, g) is called the combined contact structure of η .

2.1. Definition

A contact manifold $(M, \eta, \zeta, \phi, g)$ is said to be a Sasaki manifold if M satisfies

$$(\nabla_X \phi)Y = g(X, Y)\zeta - \eta(Y)X.$$

Since ζ is globally defined as a unit vector field, the contact manifold M admits a Lorentz metric. A Lorentz metric h is defined as $h = g - 2\eta \otimes \eta$ and h provides the following equalities [3]:

$$\begin{aligned} h(X, \zeta) &= -\eta(X), \\ h(\phi X, \phi Y) &= h(X, Y) + \eta(X)\eta(Y), \\ \partial\eta(X, Y) &= 2h(X, \phi Y). \end{aligned}$$

We call Lorentz metric h by associated Lorentz metric of M . Let us define by ∇ the Levi-Civita connection of the Lorentz metric h . Then ∇ is related to the Levi-Civita connection ∇^g of g by the following formula;

$$\nabla_X Y = \nabla_X^g Y + 2(\eta(X)\phi Y + \eta(Y)\phi X).$$

Let $(M, \eta, \zeta, \phi, g)$ be Sasaki manifold. Then the associated Lorentz metric h satisfies the following equation [5]:

$$\begin{aligned} (\nabla_{X\phi})Y &= h(X, Y)\zeta + \eta(Y)X, \\ \nabla_X \zeta &= -\phi X. \end{aligned}$$

The real vector field ζ is defined globally as the timelike killing vector field on Lorentz manifold (M, h) . As a result, the manifold $(M, \eta, \zeta, \phi, g)$ is called a Lorentz Sasaki manifold or Sasakian spacetime [3].

2.2. Definition

Let $M^3 = (M, \eta, \zeta, \phi, g)$ be a contact 3-manifold with an associated metric g . A curve $\gamma = \gamma(s) : I \rightarrow M$ parameterized by the arc length parameter is said to be a Legendre curve

if is tangent to contact distribution D of M . It is obvious that γ is the Legendre curve if and only if $\eta(\gamma') = 0$.

2.3. Definition

Let γ be a Legendre curve on M^3 . Frenet frame with components $\{V_1, V_2, V_3\}$ of the curve γ is defined as follows:

$$\begin{bmatrix} \nabla'_{\gamma} V_1 \\ \nabla'_{\gamma} V_2 \\ \nabla'_{\gamma} V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \quad (1)$$

where $V_1 = \gamma'$ and $V_2 = \zeta$. The function κ is called the curvature of γ . Namely, every Legendre curve has constant torsion 1. In addition, the following equality is defined for the Sasaki manifold M :

$$(\nabla_{X\phi})Y = g(X, Y)\zeta - \eta(Y)X, \quad X, Y \in \chi(M).$$

In particular, a curve γ parametrized by the arc length is said to be a geodesic if $\kappa = 0$. Note that if $\kappa = 0$, then automatically $\tau = 0$. More generally, a curve with constant curvature and zero torsion is called the Riemann circle. Geodesics are regarded as Riemannian circles of zero curvatures. Since a circular helix curvature and a torsion is constant, geodesics and Riemann circles can be defined as degenerate helices [10].

2.4. Definition

Legendre matrix collocation method is a method that gives an approximate solution in the form of

$$y(x) = \sum_{n=0}^N a_n P_n(x)$$

of a differential equation given as

$$\sum_{k=0}^m A_k(x) y^{(k)}(x) = g(x), \quad (-1 \leq x \leq 1).$$

It is clear that this equation is the m th order linear differential equation with variable coefficients. Since the approximate solution is found in the form of Legendre polynomials and with the help of matrices, the method is named as the Legendre matrix collocation

method. With this method, the solution is made under certain initial conditions. Also, Legendre polynomials are defined as:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} \quad (n \geq 0),$$

and the Legendre coefficients are a_n ($n = 0, 1, \dots, N$). $A_k(x)$ and $g(x)$ are defined functions [12].

3. The Characterizations of Legendre Curves

In this section the Legendre curves are characterized with the help of Frenet frame components. First, using Frenet formulas (1), the following equality is obtained:

$$\nabla_{\gamma}'^2 V_1 = -\kappa'^2 V_1 + \kappa' V_2 + \kappa V_3. \quad (2)$$

The following expression is obtained by differentiating this obtained equality (2) according to the arc length parameter:

$$\nabla_{\gamma}'^3 V_1 = -2\kappa\kappa' V_1 - \kappa'^2 \nabla_{\gamma}' V_1 + \kappa'' V_2 + \kappa' \nabla_{\gamma}' V_2 + \kappa' V_3 + \kappa \nabla_{\gamma}' V_3. \quad (3)$$

If the expressions V_3 taken from the second equation of the system (1) and V_2 taken from the first equation are used in the equation (3), the following equation is obtained:

$$V_3 = \left(\frac{1}{\kappa}\right)' \nabla_{\gamma}' V_1 + \frac{1}{\kappa} \nabla_{\gamma}'^2 V_1 + \kappa V_1. \quad (4)$$

Finally, equation (4) is differentiated and if the third equation of system (1) is used in the resulting expression, the following characterization is found:

$$\nabla_{\gamma}'^3 V_1 + \lambda_1 \nabla_{\gamma}'^2 V_1 + \lambda_2 \nabla_{\gamma}' V_1 + \lambda_3 V_1 = 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ coefficients are as follows:

$$\begin{aligned} \lambda_1 &= -2 \frac{\kappa'}{\kappa}, \\ \lambda_2 &= \kappa'^2 + 1 - \frac{\kappa''^2 - 2\kappa\kappa''}{\kappa^3}, \\ \lambda_3 &= \kappa\kappa'. \end{aligned}$$

3.1. Theorem

If γ is a unit-speed Legendre curve in Sasakian 3-Manifold, it satisfies the following differential equation:

$$\nabla_{\gamma}'^3 V_1 + \lambda_1 \nabla_{\gamma}'^2 V_1 + \lambda_2 \nabla_{\gamma}' V_1 + \lambda_3 V_1 = 0. \quad (5)$$

3.2. Conclusion

If γ unit-speed Legendre curve in Sasakian 3-Manifold is geodesic, then it is a circular helix.

3.2.1. Proof

Let γ be a geodesic curve. Then $\nabla_{\gamma}' V_1 = 0$ which gives that κ is constant i.e., γ is a circular helix.

3.3. Conclusion

If γ unit-speed Legendre curve in Sasakian 3-Manifold is a circular helix, then it satisfies the equality

$$\nabla_{\gamma}'^3 V_1 + (\kappa^2 + 1) \nabla_{\gamma}' V_1 = 0.$$

This expression is a characterization of unit-speed Legendre curves, based on the function V_1 .

The characterization of these curves, based on the functions of V_2 and V_3 is obtained as follows with the help of Frenet system (1).

3.4. Theorem

Let γ be a unit speed Legendre curve on Sasakian 3-manifold. Then satisfies the following differential equation

$$\nabla_{\gamma}'^3 V_2 + \omega_1 \nabla_{\gamma}'^2 V_2 + \omega_2 \nabla_{\gamma}' V_2 + \omega_3 V_2 = 0, \quad (6)$$

where $\omega_1, \omega_2, \omega_3$ coefficients are as follows:

$$\begin{aligned}\omega_1 &= -\frac{\kappa''}{\kappa'}, \\ \omega_2 &= -(\kappa^2 + 1), \\ \omega_3 &= 3\kappa\kappa' - \frac{\kappa'^2}{\kappa}.\end{aligned}$$

3.5. Theorem

Let γ be a unit speed Legendre curve on Sasakian 3-manifold. Then satisfies the following differential equation:

$$\nabla_\gamma^3 V_3 + \mathcal{G}_1 \nabla_\gamma^2 V_3 + \mathcal{G}_2 \nabla_\gamma V_3 + \mathcal{G}_3 V_3 = 0, \quad (7)$$

where $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ coefficients are as follows:

$$\begin{aligned}\mathcal{G}_1 &= \mathcal{G}_3 = -\frac{\kappa'}{\kappa}, \\ \mathcal{G}_2 &= \kappa^2 + 1.\end{aligned}$$

Each of these equations are linear, homogeneous differential equations 3rd order with variable coefficient.

4. The Approximate Solution for the Legendre Curve

In this section, approximate solutions of differential equations that characterize the Legendre curves are calculated. The Legendre matrix collocation method is used for the solution. The method can be applied similarly for each of the equations (5), (6) and (7) obtained depending on the functions V_1, V_2 and V_3 , respectively. In this study, the approximate solution is obtained for equation (7). First of all, the differential equation (7) is rearranged as follows:

$$\sum_{k=0}^3 A_k(x) y^{(k)}(x) = g(x), \quad -1 \leq x \leq 1, \quad (8)$$

considering

$$A_3(x) = 1, A_2(x) = A_0(x) = -\frac{\kappa'}{\kappa}, A_1(x) = \kappa^2 + 1, y^{(k)}(x) = \nabla_{\gamma}^{'k} V \text{ and } g(x) = 0.$$

It is assumed that there is an approximate solution of this equation, in the range $-1 \leq x \leq 1$, under initial conditions

$$y^{(k)}(0) = \lambda_k, \quad (k = 0, 1, 2)$$

in the form Legendre polynomials

$$y(x) = \sum_{n=0}^N a_n P_n(x). \quad (9)$$

Let $N = 3$ for convenience. Here, A_k and g are defined functions in the range $[-1, 1]$, λ_k are the appropriate constant, a_n are unknown Legendre coefficients, $P_n(x)$ are Legendre polynomials that orthogonal in the range $[-1, 1]$ relative to the value function.

4.1. Matrix Relationships for The Approximate Solution

Suppose that the function $y(x)$ can be spread over the truncated Taylor series

$$y(x) \cong \sum_{n=0}^N y_n x^n, \quad y_n = \frac{y^{(n)}(0)}{n!}$$

and the truncated Legendre series

$$y(x) \cong \sum_{n=0}^N a_n P_n(x), \quad (-1 \leq x \leq 1).$$

The approximate solution of these two different series expansions can be expressed in the form of matrix as follow;

$$[y(x)] = X(x)Y,$$

$$[y(x)] = P(x)A.$$

The matrices mentioned here are obtained as

$$P(x) = [P_0(x) \quad P_1(x) \quad P_2(x) \quad P_3(x)], \quad A = [a_0 \quad a_1 \quad a_2 \quad a_3]^T, \\ X(x) = [1 \quad x \quad x^2 \quad x^3], \quad Y = [y_0 \quad y_1 \quad y_2 \quad y_3]^T$$

for $N = 3$. Also with the help of Legendre polynomials, the components of the matrix form $P(x)$ are calculated as follow;

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2 - 1}{2}, P_3(x) = \frac{5x^3 - 3x}{2}.$$

The matrix equation

$$P(x) = X(x)D^T$$

is obtained using the equality of the expressions $[y(x)]$ obtained by two different series expansions. Here, the transition matrix is calculated as

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix}$$

for the given equation. Also, the matrix relation

$$Y = D^T A$$

is obvious between Legendre coefficient function and Taylor coefficient function. On the other hand

$$[y^{(k)}(x)] = X(x)M^k Y$$

is the matrix expression of the derivative of the approximate solution function $y(x)$, where M^0 is unit matrix and it's defined as

$$M^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.2. Matrix Relations for Differential Part

$$D(x) = \sum_{k=0}^3 A_k(x)y^{(k)}(x),$$

which is the differential part of equation (8), can be written as

$$\sum_{k=0}^3 \sum_{i=0}^3 a_{ki} x^i y^{(k)}(x)$$

with Taylor coefficients

$$a_{ki} = \frac{A_k^{(i)}(0)}{i!}.$$

Thus the matrix expression of the differential part is obtained as

$$[D(x)] = \sum_{k=0}^3 \sum_{i=0}^3 a_{ki} X(x) I_i M^k Y,$$

where I_0 is unit matrix and it's defined as

$$I_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, I_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, I_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, since

$$[g(x)] = X(x)G,$$

the matrix expression of the differential equation (8) is obtained as

$$\left\{ \sum_{k=0}^3 \sum_{i=0}^3 a_{ki} I_i M^k \right\} D^T A = G.$$

In this case, if

$$W = \sum_{k=0}^3 \sum_{i=0}^3 a_{ki} I_i M^k D^T,$$

the equation can be expressed as

$$WA = G.$$

In this equation in the matrix form, the collocation points

$$x_0 = -1, \quad x_1 = -1/3, \quad x_2 = 1/3, \quad x_3 = 1$$

are used for $x = x_i$, ($i = 0, 1, 2, 3$). Then the matrix W is calculated, the equation is written as

$$\begin{bmatrix} W \\ G \end{bmatrix}$$

in augmented matrix form.

4.3. Matrix Relations for the Initial Conditions

The matrix expression of the conditions under the initial conditions given as

$$y(0) = \lambda_0,$$

$$y'(0) = \lambda_1,$$

$$y''(0) = \lambda_2$$

is calculated as

$$U_0 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \end{bmatrix}, U_1 = \begin{bmatrix} 0 & 1 & 0 & \frac{7}{2} \end{bmatrix}, U_2 = \begin{bmatrix} 0 & 0 & 3 & 0 \end{bmatrix}.$$

4.4. The Solution

If the augmented matrix form of the matrix equation of conditions is used in the matrix $\begin{bmatrix} W \\ G \end{bmatrix}$, the following matrix is obtained:

$$\begin{bmatrix} W^* \\ G^* \end{bmatrix} = \begin{bmatrix} -\frac{\kappa'(0)}{\kappa(0)} & \kappa^2(0)+1 & -\frac{5\kappa'(0)}{2\kappa(0)} & -\frac{3}{2}(\kappa^2(0)-9) & ; & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & \frac{7}{2} & ; & \lambda_1 \\ 0 & 0 & 3 & 0 & ; & \lambda_2 \end{bmatrix}.$$

From here, with the help of the equality $A = W^{*-1}G^*$, the unknowns a_n are calculated as follows:

$$\begin{aligned}
 a_0 &= \lambda_0 + \frac{1}{6} \lambda_2, \\
 a_1 &= \frac{-7 \frac{\kappa'(0)}{\kappa(0)} \lambda_0 - 3(\kappa^2(0) - 9) \lambda_1 + 7 \frac{\kappa'(0)}{\kappa(0)} \lambda_2}{20 - 10\kappa^2(0)}, \\
 a_2 &= \frac{1}{3} \lambda_2, \\
 a_3 &= \frac{2 \frac{\kappa'(0)}{\kappa(0)} \lambda_0 - 2(\kappa^2(0) - 9) \lambda_1 - 2 \frac{\kappa'(0)}{\kappa(0)} \lambda_2}{20 - 10\kappa^2(0)}.
 \end{aligned}$$

If these a_n values are substituted in the equation (9);

$$V_3(x) = a_0 + xa_1 + \frac{3x^2 - 1}{2}a_2 + \frac{5x^3 - 3x}{2}a_3$$

is obtained. This is the coefficient function V_3 that determines Legendre curve in Sasakian 3-manifold and is also binormal of the curve.

4.5. Example

Let the first curvature of a unit-speed Legendre curve in Sasakian 3-Manifold be given as $\kappa(s) = s - 1$, $(0 \leq s \leq 2\pi)$. Find the characterization of this curve according to the binormal. Calculate the approximate value of the coefficient function determining the curve using Legendre matrix sorting method.

The first thing to note is the need to apply the conversion $x = \frac{s}{\pi} - 1$ between the range $(0 \leq s \leq 2\pi)$, where the curve is defined and the range $(-1 \leq x \leq 1)$ used in the solution method. Thus, the characterization of Legendre curve, whose first curvature $\kappa(x) = x - 1$, according to the binormal vector is obtained as follows:

$$\nabla_{\gamma'}^3 V_3 + \left(\frac{1}{1-x}\right) \nabla_{\gamma'}^2 V_3 + (x^2 - 2x + 2) \nabla_{\gamma'} V_3 + \left(\frac{1}{1-x}\right) V_3 = 0.$$

This differential equation is rearranged as follows;

$$y'''(x) + \left(\frac{1}{1-x}\right)y''(x) + (x^2 - 2x + 2)y'(x) + \left(\frac{1}{1-x}\right)y(x) = 0, \quad (10)$$

considering

$$A_1(x) = x^2 - 2x + 2, A_2(x) = A_0(x) = \frac{1}{1-x}, A_3(x) = 1, y^{(k)}(x) = \nabla_\gamma^k V, g(x) = 0.$$

Let us obtain an approximate solution for $N=3$, in the form of

$$y(x) = \sum_{n=0}^N a_n P_n(x)$$

Legendre polynomials for the Legendre curve, whose characterization is given by equation (10). The increased matrix depending on the initial conditions given as

$$y(0) = 1,$$

$$y'(0) = 0,$$

$$y''(0) = 0$$

is calculated as follows:

$$\begin{bmatrix} W^* ; G^* \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2.5 & 12 & ; & 0 \\ 1 & 0 & -0.5 & 0 & ; & 1 \\ 0 & 1 & 0 & 3.5 & ; & 0 \\ 0 & 0 & 3 & 0 & ; & 0 \end{bmatrix}.$$

This system has the solution $A = [1 \quad 0.7 \quad 0 \quad -0.2]^T$. Thus, approximate solution of differential equation (10) is obtained as $y(x) = -0.5x^3 + x + 1$. This is the approximate value under the transformation $x = \frac{s}{\pi} - 1$ of the binormal of the unit-speed Legendre curve, whose the first curvature $\kappa(s) = s - 1$, ($0 \leq s \leq 2\pi$), characterized by equation (10) in Sasakian 3-Manifold.

5. Conclusion and Suggestions

In this study, we obtain the differential equations characterizing the Legendre curves in Sasakian 3-Manifold separately for each coefficient function (tangent, normal, binormal). Then we reach to the approximate solutions of the equation related to the binormal V_3 of the curve with the Legendre matrix collocation method developed specifically. Similar solutions can be made for other coefficient functions.

Approximate solutions of the obtained differential equations can be done with different matrix methods such as Taylor, Bernstein and Hermite. In fact, the method that is more suitable for this type of curve can be determined with the help of error analysis between the two selected methods.

Frenet equation system (1) can be solved as the system for this curve type with the same method.

This study is important in order to facilitate the examination of geometric properties by combining differential geometry with differential equation solutions.

Similar studies can be adapted to different geometric equations in different spaces.

The interval $-1 \leq x \leq 1$, which has been studied in this study, has been taken as

$0 \leq s \leq 2\pi$ under the conversion of $x = \frac{s}{\pi} - 1$ for easy calculations in angular values. This

process, which we tried for the first time, did not cause any error in the method. With this logic, the interval studied in other matrix collocation methods can also be converted into a suitable different interval.

References

- [1] Baikoussis, C., Blair, D.E., On Legendre curves in contact 3-manifolds, *Geometriae Dedicata*, **49**, 1994, 135-142.
- [2] Balestro, V., Martini, H., Teixeira, R., On Legendre curves in normed planes. *Pacific Journal of Mathematics*, **297**, 2017, 1-27.
- [3] Bejancu, A., Duggal, K.L., Real hypersurfaces of indefinite Kaehler manifolds, *International Journal of Mathematics and Mathematical Sciences*, **16**, 1993, 545-556.
- [4] Belkalfa, M., Hrica, I.E., Rosca, R., Verstraelen, L., On Legendre curves in Riemannian and Lorentzian Sasaki spaces. *Soochow Journal of Mathematics*, **28**, 2002, 81-91.
- [5] Blair, D.E., *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. Springer Verlag, 1976.
- [6] Camcı, Ç., Curves Theory in Contact Geometry, *PhD thesis*, Ankara University, 2007.
- [7] Cho, J. T., Inoguchi, J.-I., Lee, J.-E., On Slant Curves in Sasakian 3-Manifolds. *Bulletin of the Australian Mathematical Society*, **74**, 2006, 359-367.
- [8] Çetin, M., Kocayigit, H., Sezer, M., Lucas collocation method to determination spherical curves in Euclidean 3-space, *Communication in Mathematical Modeling and Applications*, **3**, 2018, 44-58.
- [9] Isik, O. R., Sezer, M., Guney, Z., Bernstein series solution of linear second-order partial differential equations with mixed conditions, *Mathematical Methods in the Applied Sciences*, **37**, 5, 2014, 609-619.
- [10] Kocayigit, H., Hacısalihoğlu, H.H., Biharmonic Curves in Contact Geometry, *Communications Faculty of Sciences University of Ankara Series*, **61**, 2, 2012, 35-43.
- [11] Sezer, M., Karamete, A., Gülsu, M., Taylor Polynomial solutions of systems of linear differential equations with variable coefficients, *International Journal of Computer Mathematics*, **82**, 6, 2005, 755- 764.
- [12] Sezer, M., Gülsu, M., Solving high-order linear differential equations by a Legendre matrix method based on hybrid Legendre and Taylor polynomials, *Numerical Methods for Partial Differential Equations*, **26**, 3, 2010, 647-661.
- [13] Srivastava, K., Sood , K., Srivastava, S. K., Characterization of Legendre curves in quasi-Sasakian pseudo-metric 3-manifolds, *arXiv: General Mathematics*, 2019.
- [14] Yano, K., Kon, M. Structures on manifolds, *Series in pure mathematics*, **3**, 1984.
- [15] Yıldırım, A., Tubular surface around a Legendre curve in BCV spaces, *New Trends in Mathematical Sciences*, **4**, 2, 2016, 61-71.